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On the intersections of A_1 subgroups in the exceptional simple Lie groups E_6, E_7 and E_8

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Abstract. We determine reductions of irreducible representations of E_6, E_7 and E_8 according to the chain of subgroups $E_k \supset A_1 \supset T_k$, $k = 6, 7, 8$, for all A_1 subgroups of E_k. Here A_1 is either SU(2) or O(3) and T_6, T_7, T_8 are respectively the tetrahedral, octahedral and icosahedral groups. The result of the article is the list of coinciding branching rules for $E_k \supset T_k$ proceeding via different subgroups A_1.

1. Introduction

The purpose of this paper is to point out yet another aspect of the curious relation between the finite subgroups of the Lie group SU(2) and the exceptional simple Lie groups [1-4].

It is well known that one can associate with a finite subgroup of SU(2) a Coxeter-Dynkin diagram. Minimal resolution of Kleinian singularities provides a relation among the following groups and the diagrams with equal length roots:

Z_{r+1} ——— A_r

D_{r-2} ——— D_r

T ——— E_6

O ——— E_7

I ——— E_8

where on the left-hand side are the cyclic group Z_r of order r, the (double) dihedral group D_r of order $4r$, the (double) tetrahedral group T, the (double) octahedral group O, and the (double) icosahedral group I. On the right-hand side are the compact simple Lie groups of equal root lengths.

We consider the chains of two subgroups

$$E_k \supset A_1 \supset T_k \qquad k = 6, 7, 8 \tag{1}$$

where E_6, E_7 and E_8 are the compact exceptional simple Lie groups, A_1 is one of the locally isomorphic simple Lie groups SU(2) or O(3). It is also well known that the finite subgroups of SU(2) are the so-called double finite groups while those in O(3)

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are not double. For simplicity of notation T_6, T_7, T_8 denote respectively the (double or not) tetrahedral, octahedral and icosahedral subgroup of the appropriate A_1.

The groups E_k contain many subgroups A_1 which are not conjugate under E_k. Among those there are two unique ones, up to E_k conjugacy, in each E_k (in fact in any simple Lie group), called the principal and subprincipal A_1. It was conjectured [2] that the intersection of A_1(principal) with A_1(subprincipal) in E_k is the finite group T_k. Assuming this conjecture it would follow in particular that the two chains of subgroups

$$E_k \supset A_1(\text{principal}) \supset T_k \tag{2a}$$

$$E_k \supset A_1(\text{subprincipal}) \supset T_k \tag{2b}$$

yield the same branching rule

$$\Phi(E_k) \supset \Phi(T_k) \tag{3}$$

where $\Phi(G)$ is any representation of the group G. Since one could have isomorphic finite subgroups in E_k conjugate under the action of the general linear group but not E_k conjugate, the coincidence of the branching rules is a strong necessary condition, but not a sufficient condition, for proving that the intersection of the two A_1 in (2) is T_k. Conversely, the absence of the coincidence rules out the conjecture.

The problem we consider here is the branching rule for the chains (2) where the principal and subprincipal A_1 are replaced by any subgroup of type A_1 of E_k. Naturally, only pairwise non-conjugate groups A_1 are considered. It turns out that the coincidences of the branching rules are numerous, including several not suggested by Slodowy. The main result of the article is the list of these coincidences for all A_1 in E_6, E_7 and E_8. In particular, it turns out that there are 7, 5 and 5 chains of subgroups with different A_1 groups, containing (2*a*) and (2*b*), respectively in E_8, E_7 and E_6 giving the same branching rules (3).

In order to verify that the branching rules coincide for all representations of E_k, $k=6, 7, 8$, it suffices that we calculate them for the three 'tip' representations shown on figure 1. Indeed, it is well known that any other irreducible representation of finite dimension is built by alternation, symmetrization and tensor product of the tip representations.

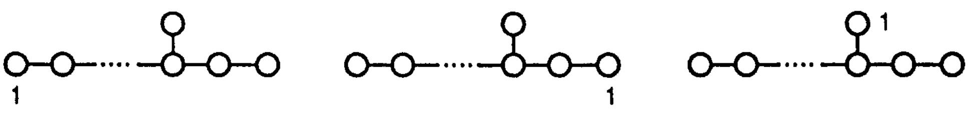

Figure 1. The tip representations of the exceptional simple Lie groups E_k, $k=6, 7, 8$. The non-zero component of the highest weight is attached to the corresponding simple roots.

2. The reduction of E_k to A_1 subgroups.

The list of non-conjugate A_1 subgroups of E_k is provided by Dynkin [5]. The A_1 of type O(3) is called integral in [5], while the non-integral stands for SU(2). With a few minor corrections that is the list we use. For our purposes the groups A_1 on Dynkin's list are numbered in table 1 and specified by their $E_k \to A_1$ projection matrices. The projection matrices refer to a numbering of simple roots shown in figure 2. The subgroups A_1(principal) and A_1(subprincipal) are easily singled out in table 1: the corresponding projection matrices have the largest and second largest values of matrix elements. It is more revealing to introduce A_1(principal) as the group whose simple

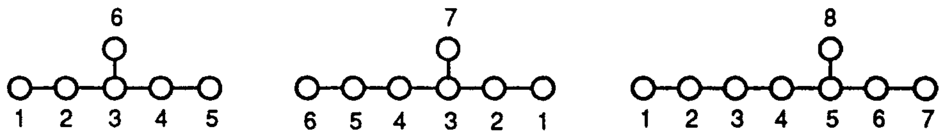

Figure 2. Numbering of simple roots of E_6, E_7, E_8 (Dynkin's numbering).

root is the image of the application of the corresponding projection matrix to every simple root of E_k. Similarly, A_1(subprincipal) is the group whose simple root is the image of the application of the corresponding projection matrix to all but one simple root of E_k: the root at the base of the fork of the Coxeter–Dynkin diagram is mapped to zero by the projection matrix of A_1(subprincipal) in our set-up.

We proceed in two steps. First we find the reductions

$$E_k \supset A_1 \qquad k = 6, 7, 8 \tag{4}$$

for the three tip representations and for every A_1 on the list. Naturally, that yields no coinciding branching rules because any two such subgroups A_1 would not then be on the list simultaneously. The second step is the reduction of each A_1 to the finite group T_k.

Only a few of the inclusions (4) are maximal. More precisely, A_1(principal) is maximal in E_6, E_7 and E_8; A_1(subprincipal) is maximal in E_7 and E_8; there is one more A_1 maximal in E_8. For the maximal inclusions the branching rules $E_k \supset A_1$ for the tip representations can be found in [6]. It is practical to calculate the branching rules for the tip representations directly all at once rather than identifying a chain of maximal subgroups. Some of those chains would be quite long.

The $E_k \to A_1$ projection matrix [7] can be used as an efficient way to compute the branching rules $E_k \supset A_1$ for the tip representations. In general, it is a $k \times 1$ matrix row transforming an E_k weight written as a $1 \times k$ matrix column into an A_1 weight which is just an integer. The projection matrix specifies an inclusion up to a linear equivalence. They are listed in table 1 relative to the basis of fundamental weights.

In spite of the large number of weights one has to project (the largest tip representation of E_8 is of dimension 147 250), this method is quite practical when one considers separately reductions of E_k-Weyl-group orbits of weights to the Weyl group orbits of A_1 weights. In the largest orbit one then encounters, there are only 17 280 weights. The structure of a Weyl group orbit of A_1 is extremely simple: it either contains just the zero weight or it consists of non-zero weight and its negative. Therefore it is easy to sort out the projected weights into Weyl group orbits of A_1.

The results from the reduction of each E_k-Weyl group orbit are then put together with the multiplicities appropriate for the tip representations [6, 8]. Even on a small desktop computer, this requires only a few minutes of computer time. Our final computation was done using the software simpLie [9] on Macintosh Plus. The resulting branching rules $E_k \supset A_1$ are not shown here. They are used as the input for the second step of our computation.

3. The reduction of A_1 to T_k subgroups

The next step is the reduction of every irreducible representation $\Gamma(A_1)$ of A_1, which occurs in the $E_k \supset A_1$ branching rules for the three tip representations, to the representations $\Gamma(T_k)$ of T_k. An irreducible representation $\Gamma_j(A_1)$ is labelled by its highest weight

Table 1. Projection matrices for the reduction of E_k weights to A_1 weights relative to the Dynkin numbering and the basis of fundamental weights. Each numbered line is one projection matrix of E_k; $k=6, 7, 8$ (a, b, c).

# 1	2	4	6	4	2	4
# 2	4	6	8	6	4	4
# 3	4	8	12	8	4	6
# 4	6	10	14	10	6	8
# 5	6	12	18	12	6	10
# 6	8	14	20	14	8	10
# 7	10	18	26	18	10	14
# 8	12	22	30	22	12	16
# 9	16	30	42	30	16	22
#10	1	2	3	2	1	2
#11	2	3	4	3	2	2
#12	2	4	6	4	2	3
#13	3	5	7	5	3	4
#14	3	6	8	6	3	4
#15	4	7	10	7	4	5
#16	4	7	10	7	4	6
#17	4	8	11	8	4	6
#18	6	11	15	11	6	8
#19	7	13	18	13	7	10
#20	8	14	19	14	8	10

(*a*)

# 1	2	4	6	5	4	3	3
# 2	4	6	8	6	4	2	4
# 3	4	8	12	9	6	3	7
# 4	4	8	12	10	8	4	6
# 5	6	10	14	11	8	5	7
# 6	6	12	16	12	8	4	8
# 7	6	12	18	15	10	5	9
# 8	8	14	20	16	12	6	10
# 9	8	16	24	18	12	6	12
#10	10	18	24	18	12	6	12
#11	10	18	26	21	14	7	13
#12	10	18	26	21	16	9	13
#13	10	20	28	22	16	8	14
#14	10	20	30	23	16	9	15
#15	12	24	36	28	20	10	18
#16	14	26	36	28	20	10	18
#17	14	26	38	29	20	11	19
#18	16	30	44	34	24	12	22
#19	18	34	50	39	28	15	25
#20	22	42	60	46	32	16	30
#21	22	42	60	47	32	17	31
#22	26	50	72	57	40	21	37
#23	34	66	96	75	52	27	49
#24	2	3	4	3	2	1	2
#25	2	4	6	5	4	2	3
#26	3	6	8	6	4	2	4
#27	3	6	9	7	5	3	5
#28	4	7	10	8	6	3	5
#29	4	8	12	9	6	3	6
#30	5	10	14	11	8	4	7
#31	6	10	14	11	8	4	7
#32	6	11	16	12	8	4	8
#33	6	11	16	13	9	5	8
#34	6	12	17	13	9	5	9
#35	6	12	18	14	10	5	9
#36	8	15	22	17	12	6	11
#37	10	18	25	19	13	7	13
#38	10	18	26	20	14	7	13
#39	10	19	28	22	16	8	14
#40	10	19	28	22	16	9	14
#41	10	20	29	23	16	9	15
#42	14	26	37	29	20	10	19
#43	14	26	37	29	20	11	19
#44	18	34	49	39	28	15	25

(*b*)

# 1	4	6	8	10	12	8	4	6
# 2	4	8	12	16	20	14	8	10
# 3	6	12	16	20	24	16	8	12
# 4	6	12	18	24	30	20	10	16
# 5	8	14	20	26	32	22	12	16
# 6	8	16	24	30	36	24	12	18
# 7	10	18	24	30	36	24	12	18
# 8	10	18	26	34	42	28	14	22
# 9	10	20	28	36	44	30	16	22
#10	10	20	30	40	48	32	16	24
#11	12	24	36	46	56	38	20	28
#12	14	26	36	46	56	38	20	28
#13	14	26	38	50	60	40	20	30
#14	16	30	44	56	68	46	24	34
#15	16	30	44	58	72	48	24	36
#16	18	34	50	66	80	54	28	40
#17	18	36	52	68	84	56	28	42
#18	22	42	60	76	92	62	32	46
#19	22	42	60	78	96	64	32	48
#20	22	44	64	84	104	70	36	52
#21	26	50	72	94	116	78	40	58
#22	28	54	80	104	128	86	44	64
#23	34	66	96	124	152	102	52	76
#24	38	74	108	142	174	118	60	88
#25	46	90	132	172	210	142	72	106
#26	58	114	168	220	270	182	92	136
#27	2	3	4	5	6	4	2	3
#28	2	4	6	8	10	7	4	5
#29	3	6	8	10	12	8	4	6
#30	3	6	9	12	15	10	5	8
#31	4	7	10	13	16	11	6	8
#32	4	8	12	15	18	12	6	9
#33	4	8	12	16	20	14	7	10
#34	5	10	14	18	22	15	8	11
#35	6	10	14	18	22	15	8	11
#36	5	10	15	20	24	16	8	12
#37	6	11	16	20	24	16	8	12
#38	6	11	16	21	26	18	9	13
#39	6	12	17	22	27	18	9	14
#40	6	12	18	23	28	19	10	14
#41	6	12	18	24	30	20	10	15
#42	7	14	21	28	34	23	12	17
#43	8	15	22	28	34	23	12	17
#44	8	15	22	29	36	24	12	18
#45	8	16	24	31	38	26	13	19
#46	10	18	25	32	39	26	13	20
#47	10	18	26	33	40	27	14	20
#48	9	18	26	34	42	28	14	21
#49	10	18	26	34	42	28	14	21
#50	10	19	28	36	44	30	15	22
#51	10	19	28	36	44	30	16	22
#52	10	19	28	37	46	31	16	23
#53	10	20	29	38	46	31	16	23
#54	10	20	29	38	47	32	16	24
#55	10	20	30	39	48	32	16	24
#56	12	24	36	47	58	39	20	29
#57	14	26	37	48	58	39	20	29
#58	14	26	37	48	59	40	20	30
#59	14	26	38	49	60	40	20	30
#60	14	27	40	52	64	43	22	32
#61	15	30	44	57	70	47	24	35
#62	16	30	44	57	70	47	24	35
#63	18	34	49	64	79	54	28	40
#64	18	34	50	65	80	54	28	40
#65	22	42	60	77	94	63	32	47
#66	22	42	60	78	95	64	32	48
#67	22	43	64	84	103	70	36	52
#68	26	50	72	94	115	78	40	58
#69	34	66	96	124	151	102	52	76

(*c*)

which has just one integer component J (twice the angular momentum). The irreducible representations $\Gamma(\mathrm{T}_k)$ of the finite groups are listed in table 2. They are numbered from 1 up; Γ_1 denotes always the identity representation. Further properties of these repesentations are found, for example, in [10, 11].

The branching rules

$$\Gamma_J(\mathrm{A}_1) \supset \Gamma(\mathrm{T}_k) = \bigoplus_i m_i(J)\Gamma_i(\mathrm{T}_k) \qquad k = 6, 7, 8, \tag{5}$$

for a given irreducible representation $\Gamma_J(\mathrm{A}_1)$ are conveniently provided for all $\Gamma_J(\mathrm{A}_1)$, $J = 0, 1, 2, \ldots$, at once by the generating functions of [10] and [11]. The relevant generating functions are reproduced in table 3.

Let us explain how the branching rules (5) are found using table 3. The representation $\Gamma(\mathrm{T}_k)$ of (5) is a direct sum of the irreducible representations $\Gamma_i(\mathrm{T}_k)$ of T_k each

Table 2. Dimensions of the irreducible representations Γ of the finite groups T_k, $k = 6, 7, 8$.

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7	Γ_8	Γ_9
T_6	1	1	1	3	2	2	2	—	—
T_7	1	1	2	3	3	2	2	4	—
T_8	1	3	3	4	5	2	2	4	6

Table 3. Numerators and denominators of the generating functions $F(\mathrm{A}_1, \Gamma_i(\mathrm{T}_k))$.

	T_6
Γ_1	$1+x^{12}$
Γ_2	x^4+x^8
Γ_3	x^4+x^8
Γ_4	$x^2+x^4+2x^6+x^8+x^{10}$
Γ_5	$x+x^5+x^7+x^{11}$
Γ_6	$x^3+x^5+x^7+x^9$
Γ_7	$x^3+x^5+x^7+x^9$
D	$(1-x^6)(1-x^8)$

	T_7
Γ_1	$1+x^{18}$
Γ_2	x^6+x^{12}
Γ_3	$x^4+x^8+x^{10}+x^{14}$
Γ_4	$x^2+x^6+x^8+x^{10}+x^{12}+x^{16}$
Γ_5	$x^4+x^6+x^8+x^{10}+x^{12}+x^{14}$
Γ_6	$x+x^7+x^{11}+x^{17}$
Γ_7	$x^5+x^7+x^{11}+x^{13}$
Γ_8	$x^3+x^5+x^7+2x^9+x^{11}+x^{13}+x^{15}$
D	$(1-x^8)(1-x^{12})$

	T_8
Γ_1	$1+x^{30}$
Γ_2	$x^2+x^{10}+x^{12}+x^{18}+x^{20}+x^{28}$
Γ_3	$x^6+x^{10}+x^{14}+x^{16}+x^{20}+x^{24}$
Γ_4	$x^6+x^8+x^{12}+x^{14}+x^{16}+x^{18}+x^{22}+x^{24}$
Γ_5	$x^4+x^8+x^{10}+x^{12}+x^{14}+x^{16}+x^{18}+x^{20}+x^{22}+x^{26}$
Γ_6	$x+x^{11}+x^{19}+x^{29}$
Γ_7	$x^7+x^{13}+x^{17}+x^{23}$
Γ_8	$x^3+x^9+x^{11}+x^{13}+x^{17}+x^{19}+x^{21}+x^{27}$
Γ_9	$x^5+x^7+x^9+x^{11}+x^{13}+2x^{15}+x^{17}+x^{19}+x^{21}+x^{23}+x^{25}$
D	$(1-x^{12})(1-x^{20})$

with a multiplicity m_i. One column of the table refers to one finite group. Each entry in a column is a polynomial in x. For each irreducible representation Γ_i of T_k the column gives one generating function $F(A_1, \Gamma_i(T_k))$. The generating function consists of a numerator, which is different for each Γ_i, listed on the line labelled by Γ_i, and the denominator D which is common for $\Gamma_i(T_k)$; D is shown on the last line of the column. Thus for example,

$$F(A_1, \Gamma_7(T_6)) = \frac{x^3+x^5+x^7+x^9}{(1-x^6)(1-x^8)} = x^3+x^5+x^7+x^8+2x^9+\ldots+m_7(J)x^J+\ldots \tag{6}$$

is the generating function for the multiplicities $m_7(J)$ of the representation $\Gamma_7(T_6)$ in the reduction of $\Gamma_J(A_1)$ for any J. In particular, the presence of the term $2x^9$ in the power series indicates that the representation $\Gamma_7(T_6)$ appears with multiplicity 2 in the branching rule for the ten-dimensional irreducible representation $\Gamma_9(A_1)$ of A_1.

4. The reduction of E_k to T_k

Combining the two steps of the reduction described in sections 2 and 3, one gets the desired branching rule (3) for each A_1 subgroup of E_k. The results are summarized in table 4. The table has two parts for E_6 and three parts for E_7 and E_8. The two tip representations of E_6 of the same dimension 27 are related by the outer automorphism of E_6 and the branching rules coincide for them, hence only one of the two is shown. Each part splits further into three subtables with common numbering of lines. Each subtable provides the branching rules for one of the tip representations of figure 1. The numbers in the left column identify the intermediate subgroup A_1 to which the branching rule belong. The A_1 subgroup is otherwise specified in table 1 where its projection matrix is labelled by the same number.

Table 4-E_6. The multiplicities of the irreducible representations of the tetrahedral group in the reduction of the representations (100000) and (000001) of E_6 to T_6. A numbered line contains the two branching rules proceeding via the subgroup A_1 of the same number as in table 1.

	(100000)							(000001)						
	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7
#1	9	0	0	6	0	0	0	16	1	1	20	0	0	0
#2	1	1	1	8	0	0	0	14	8	8	16	0	0	0
#3	3	3	3	6	0	0	0	4	7	7	20	0	0	0
#4	3	3	3	6	0	0	0	10	4	4	20	0	0	0
#5	9	0	0	6	0	0	0	16	1	1	20	0	0	0
#6	3	3	3	6	0	0	0	4	7	7	20	0	0	0
#7	3	3	3	6	0	0	0	4	7	7	20	0	0	0
#8	3	3	3	6	0	0	0	4	7	7	20	0	0	0
#9	3	3	3	6	0	0	0	4	7	7	20	0	0	0
#10	15	0	0	0	6	0	0	35	0	0	1	20	0	0
#11	8	0	0	1	8	0	0	22	0	0	8	16	0	0
#12	6	0	0	3	6	0	0	11	0	0	9	16	2	2
#13	3	0	0	4	4	1	1	9	1	1	9	8	6	6
#14	2	0	0	3	4	2	2	4	3	3	12	8	4	4
#15	1	1	1	4	2	2	2	3	4	4	9	8	6	6
#16	6	1	1	1	0	4	4	12	5	5	8	0	8	8
#17	2	2	2	3	2	2	2	5	3	3	9	4	8	8
#18	1	1	1	4	2	2	2	3	4	4	9	8	6	6
#19	2	0	0	3	4	2	2	4	3	3	12	8	4	4
#20	2	2	2	3	2	2	2	5	3	3	9	4	8	8

Table 4-E_7. The multiplicities of the irreducible representations of the octahedral group in the reduction of the representations (1000000), (0000010) and (0000001) of E_7 to T_7. A numbered line contains the three branching rules proceeding via the subgroup A_1 of the same number as in table 1.

	(0000010)								(1000000)								(0000001)							
	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7	Γ_8	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7	Γ_8	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7	Γ_8
# 1	0	0	0	0	0	26	0	1	52	0	0	27	0	0	0	0	0	0	0	0	0	300	0	78
# 2	20	0	0	12	0	0	0	0	35	0	1	31	1	0	0	0	152	0	32	200	32	0	0	0
# 3	0	0	0	0	0	14	0	7	14	0	7	28	7	0	0	0	0	0	0	0	0	127	35	147
# 4	4	0	2	14	2	0	0	0	17	0	10	22	10	0	0	0	72	16	76	132	92	0	0	0
# 5	0	0	0	0	0	7	1	10	21	1	15	11	16	0	0	0	0	0	0	0	0	82	62	156
# 6	8	0	6	6	6	0	0	0	9	2	13	17	15	0	0	0	34	34	74	124	108	0	0	0
# 7	0	0	0	0	0	5	3	10	3	5	10	20	15	0	0	0	0	0	0	0	0	80	76	150
# 8	6	2	6	4	8	0	0	0	10	7	10	15	17	0	0	0	36	44	68	120	112	0	0	0
# 9	0	4	2	10	6	0	0	0	6	5	13	14	18	0	0	0	42	34	82	108	116	0	0	0
#10	14	6	0	6	6	0	0	0	21	14	1	17	15	0	0	0	76	76	32	124	108	0	0	0
#11	0	0	0	0	0	8	6	7	6	8	7	20	15	0	0	0	0	0	0	0	0	83	79	147
#12	0	0	0	0	0	1	7	10	21	1	15	11	16	0	0	0	0	0	0	0	0	62	82	156
#13	6	2	6	4	8	0	0	0	7	4	13	15	17	0	0	0	30	38	74	120	112	0	0	0
#14	0	0	0	0	0	3	5	10	3	5	10	20	15	0	0	0	0	0	0	0	0	76	80	150
#15	0	4	2	10	6	0	0	0	9	8	10	14	18	0	0	0	48	40	76	108	116	0	0	0
#16	8	0	6	6	6	0	0	0	9	2	13	17	15	0	0	0	34	34	74	124	108	0	0	0
#17	0	0	0	0	0	5	3	10	3	5	10	20	15	0	0	0	0	0	0	0	0	80	76	150
#18	6	2	6	4	8	0	0	0	7	4	13	15	17	0	0	0	30	38	74	120	112	0	0	0
#19	0	0	0	0	0	5	3	10	3	5	10	20	15	0	0	0	0	0	0	0	0	80	76	150
#20	8	0	6	6	6	0	0	0	9	2	13	17	15	0	0	0	34	34	74	124	108	0	0	0
#21	0	0	0	0	0	5	3	10	3	5	10	20	15	0	0	0	0	0	0	0	0	80	76	150
#22	0	0	0	0	0	3	5	10	3	5	10	20	15	0	0	0	0	0	0	0	0	76	80	150
#23	0	0	0	0	0	5	3	10	3	5	10	20	15	0	0	0	0	0	0	0	0	80	76	150
#24	32	0	0	0	0	12	0	0	66	0	0	1	0	32	0	0	352	0	0	32	0	232	0	0
#25	18	0	0	2	0	16	0	0	39	0	0	10	0	32	0	0	188	0	0	92	0	192	0	16
#26	14	0	0	6	0	12	0	0	24	0	0	15	0	28	0	2	112	0	6	102	6	152	0	40
#27	6	0	0	6	0	14	0	1	21	0	0	16	0	20	0	6	76	0	20	96	20	121	1	51
#28	8	0	0	8	0	8	0	2	16	0	1	16	1	16	0	8	64	0	24	88	24	88	8	68
#29	6	0	0	6	0	8	0	4	9	0	3	15	3	16	0	8	46	2	34	80	36	76	12	68
#30	4	0	2	6	2	4	0	4	6	0	6	11	6	10	2	10	26	10	38	64	48	52	28	76
#31	14	0	2	0	2	0	0	8	24	1	7	2	8	0	0	16	84	16	44	32	60	16	16	96
#32	6	0	4	2	4	4	0	4	9	1	7	7	8	6	2	12	30	14	40	54	54	40	32	80
#33	2	0	2	4	2	3	1	6	6	1	7	8	8	8	4	10	20	16	40	60	56	42	34	74
#34	0	0	4	4	4	5	1	3	6	2	5	10	7	4	4	12	16	16	40	56	56	45	37	75
#35	2	0	2	4	2	2	2	6	4	3	4	11	7	4	4	12	24	20	36	60	56	36	36	76
#36	2	2	2	4	4	2	2	4	3	3	6	8	9	6	6	10	18	18	38	56	56	40	40	76
#37	4	4	0	4	4	6	2	2	10	6	1	10	7	8	8	8	32	32	24	56	56	52	44	68
#38	4	2	0	4	2	4	4	4	5	4	3	11	7	8	8	8	26	22	34	60	56	44	44	68
#39	6	0	4	2	4	0	4	4	9	1	7	7	8	2	6	12	30	14	40	54	54	32	40	80
#40	0	2	2	2	4	1	3	6	6	1	7	8	8	8	4	10	16	20	40	56	60	34	42	74
#41	0	2	2	2	4	2	2	6	4	3	4	11	7	4	4	12	20	24	36	56	60	36	36	76
#42	0	0	4	4	4	5	1	3	6	2	5	10	7	4	4	12	16	16	40	56	56	45	37	75
#43	2	0	2	4	2	2	2	6	4	3	4	11	7	4	4	12	24	20	36	60	56	36	36	76
#44	2	0	2	4	2	2	2	6	4	3	4	11	7	4	4	12	24	20	36	60	56	36	36	76

Table 4-E_8. The multiplicities of the irreducible representations of the icosahedral group in the reduction of the representations (10000000), (00000010) and (00000001) of E_8 to T_8. A numbered line contains the three branching rules proceeding via the subgroup A, of the same number as in table 1.

	(10000000)									(00000010)								
	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7	Γ_8	Γ_9	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7	Γ_8	Γ_9
# 1	78	55	0	0	1	0	0	0	0	705	835	0	0	133	0	0	0	0
# 2	28	50	0	0	14	0	0	0	0	301	518	63	64	315	0	0	0	0
# 3	28	27	2	2	25	0	0	0	0	157	351	131	158	328	0	0	0	0
# 4	8	28	8	8	20	0	0	0	0	119	252	162	216	330	0	0	0	0
# 5	24	11	11	12	22	0	0	0	0	112	208	208	245	307	0	0	0	0
# 6	6	18	13	16	17	0	0	0	0	73	192	182	250	336	0	0	0	0
# 7	52	2	27	26	1	0	0	0	0	352	131	351	353	133	0	0	0	0
# 8	8	15	15	20	14	0	0	0	0	91	183	188	274	315	0	0	0	0
# 9	14	5	10	16	25	0	0	0	0	65	190	200	250	328	0	0	0	0
#10	0	14	14	16	20	0	0	0	0	75	185	185	260	330	0	0	0	0
#11	9	13	18	19	14	0	0	0	0	94	182	192	271	315	0	0	0	0
#12	21	11	11	9	25	0	0	0	0	91	208	208	224	328	0	0	0	0
#13	3	18	13	13	20	0	0	0	0	79	192	182	256	330	0	0	0	0
#14	14	10	5	16	25	0	0	0	0	65	200	190	250	328	0	0	0	0
#15	2	15	15	14	20	0	0	0	0	76	188	183	259	330	0	0	0	0
#16	2	15	15	14	20	0	0	0	0	76	183	188	259	330	0	0	0	0
#17	3	14	14	19	17	0	0	0	0	69	185	185	254	336	0	0	0	0
#18	14	5	10	16	25	0	0	0	0	65	190	200	250	328	0	0	0	0
#19	0	14	14	16	20	0	0	0	0	75	185	185	260	330	0	0	0	0
#20	3	18	13	13	20	0	0	0	0	79	192	182	256	330	0	0	0	0
#21	2	15	15	14	20	0	0	0	0	76	183	188	259	330	0	0	0	0
#22	0	14	14	16	20	0	0	0	0	75	185	185	260	330	0	0	0	0
#23	0	14	14	16	20	0	0	0	0	75	185	185	260	330	0	0	0	0
#24	0	14	14	16	20	0	0	0	0	75	185	185	260	330	0	0	0	0
#25	0	14	14	16	20	0	0	0	0	75	185	185	260	330	0	0	0	0
#26	0	14	14	16	20	0	0	0	0	75	185	185	260	330	0	0	0	0
#27	133	1	0	0	0	56	0	0	0	1540	133	0	0	0	968	0	0	0
#28	78	14	0	0	0	64	0	0	0	819	377	0	0	1	832	0	64	0
#29	55	27	0	0	0	52	0	2	0	508	432	0	0	27	652	0	158	0
#30	36	28	0	0	0	48	0	8	0	363	414	0	0	70	504	0	216	8
#31	35	32	0	0	1	32	0	12	0	287	385	1	1	98	384	0	244	32
#32	24	27	0	0	3	32	0	16	0	207	336	10	10	134	336	0	240	48
#33	17	28	0	0	7	28	0	14	0	175	294	21	21	147	280	2	238	70
#34	17	23	0	0	10	18	0	16	2	120	241	41	42	161	206	16	208	110
#35	55	1	1	1	11	0	0	32	0	396	66	66	67	179	32	32	352	64
#36	10	20	0	0	10	20	0	16	4	105	215	55	60	160	180	20	200	120
#37	24	10	1	1	15	14	0	18	2	137	159	55	65	180	146	32	206	126
#38	13	15	1	1	11	14	0	16	6	84	171	64	79	170	146	34	174	144
#39	9	16	2	2	13	14	0	12	6	73	163	72	89	161	126	40	170	154
#40	11	16	3	3	8	8	0	16	8	76	154	80	103	153	104	40	168	160
#41	6	15	5	5	10	10	0	14	6	58	138	78	102	165	110	50	152	168
#42	10	6	6	6	10	4	4	16	8	62	101	101	123	159	64	64	152	176
#43	9	8	7	8	10	8	2	8	10	40	105	97	122	161	80	62	128	190
#44	4	9	5	6	10	8	4	8	12	38	105	98	127	160	72	60	132	188
#45	3	7	5	8	13	6	4	8	10	34	100	96	128	161	66	62	132	192
#46	21	3	15	14	1	14	6	0	12	123	81	141	165	98	118	88	66	210
#47	15	9	9	9	3	8	8	0	16	73	106	106	144	134	88	88	80	208
#48	3	6	6	9	9	4	4	10	12	29	99	99	128	164	66	66	126	192
#49	6	8	9	11	7	10	6	2	12	52	93	99	144	147	80	76	112	196
#50	3	8	5	7	10	4	6	6	14	35	102	98	130	160	60	66	126	194
#51	17	2	2	8	15	0	0	4	16	81	79	79	121	180	48	48	124	208
#52	6	3	6	8	11	4	2	8	14	37	92	96	126	170	68	62	124	194
#53	3	6	6	8	13	4	4	6	12	33	96	99	129	161	64	62	130	194
#54	6	5	9	8	8	0	4	12	12	45	98	105	134	153	52	64	140	188
#55	3	9	8	8	10	0	4	8	12	34	99	93	126	165	64	68	124	196
#56	6	6	6	11	10	4	4	6	12	33	96	99	129	161	70	68	124	194
#57	6	8	7	5	13	8	2	8	10	40	105	97	122	161	74	56	134	190
#58	6	9	5	8	8	4	0	12	12	45	105	98	134	153	64	52	140	188
#59	4	10	8	7	10	4	2	10	10	37	101	94	123	165	70	68	130	190
#60	2	6	6	8	10	6	6	8	12	33	99	99	132	160	64	64	128	192
#61	6	6	3	8	11	?	4	8	14	37	96	92	126	170	62	68	124	194
#62	3	7	5	8	13	6	4	8	10	34	100	96	128	161	66	62	132	192
#63	10	9	9	4	8	0	0	8	16	54	106	106	125	153	48	48	120	208
#64	3	8	9	8	10	4	0	8	12	34	93	99	126	165	68	64	124	196
#65	3	6	6	8	13	4	4	6	12	33	96	99	129	161	64	62	130	194
#66	3	9	8	8	10	0	4	8	12	34	99	93	126	165	64	68	124	196
#67	6	9	5	8	8	4	0	12	12	45	105	98	134	153	64	52	140	188
#68	3	8	9	8	10	4	0	8	12	34	93	99	126	165	68	64	124	196
#69	3	9	8	8	10	0	4	8	12	34	99	93	126	165	64	68	124	196

Table 4-E_8. (continued)

	(00000001)								
	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7	Γ_8	Γ_9
# 1	16720	25707	912	912	9405	0	0	0	0
# 2	6580	13894	4859	5824	12223	0	0	0	0
# 3	3586	9585	6645	8556	12150	0	0	0	0
# 4	2812	7931	7126	9448	12295	0	0	0	0
# 5	2531	7556	7556	9737	12087	0	0	0	0
# 6	2476	7366	7346	9802	12286	0	0	0	0
# 7	6331	6645	9585	11301	9405	0	0	0	0
# 8	2525	7344	7354	9879	12223	0	0	0	0
# 9	2340	7482	7502	9802	12150	0	0	0	0
#10	2455	7350	7350	9805	12295	0	0	0	0
#11	2539	7346	7366	9865	12223	0	0	0	0
#12	2468	7556	7556	9674	12150	0	0	0	0
#13	2467	7366	7346	9793	12295	0	0	0	0
#14	2340	7502	7482	9802	12150	0	0	0	0
#15	2453	7354	7344	9807	12295	0	0	0	0
#16	2453	7344	7354	9807	12295	0	0	0	0
#17	2464	7350	7350	9814	12286	0	0	0	0
#18	2340	7482	7502	9802	12150	0	0	0	0
#19	2455	7350	7350	9805	12295	0	0	0	0
#20	2467	7366	7346	9793	12295	0	0	0	0
#21	2453	7344	7354	9807	12295	0	0	0	0
#22	2455	7350	7350	9805	12295	0	0	0	0
#23	2455	7350	7350	9805	12295	0	0	0	0
#24	2455	7350	7350	9805	12295	0	0	0	0
#25	2455	7350	7350	9805	12295	0	0	0	0
#26	2455	7350	7350	9805	12295	0	0	0	0
#27	42427	10317	0	0	0	35112	0	912	0
#28	20410	15925	0	0	1093	24960	0	5824	64
#29	12337	14850	78	78	3189	17712	0	8478	756
#30	8496	12762	447	448	4747	13176	56	9000	1856
#31	6346	10869	981	1013	5486	10152	264	8724	3024
#32	4898	9228	1568	1706	5908	8240	560	8096	3936
#33	3938	7945	2037	2331	6034	6832	914	7476	4746
#34	2763	6171	2717	3354	6107	4950	1516	6448	5858
#35	6347	3478	3478	3875	6187	2176	2176	8544	5120
#36	2385	5560	2960	3745	6145	4340	1740	6060	6200
#37	2330	4883	3089	3983	6240	3794	1980	5882	6466
#38	1878	4716	3276	4264	6148	3502	2052	5552	6714
#39	1686	4445	3375	4413	6116	3236	2176	5394	6912
#40	1617	4248	3468	4585	6109	2976	2216	5280	7016
#41	1469	4077	3517	4638	6115	2868	2308	5176	7136
#42	1347	3714	3714	4861	6115	2464	2464	5016	7280
#43	1251	3709	3667	4866	6107	2508	2446	4936	7370
#44	1239	3702	3672	4891	6145	2488	2448	4916	7344
#45	1216	3679	3671	4883	6116	2474	2466	4922	7384
#46	2039	3531	4012	5320	5486	3108	2784	4200	7548
#47	1522	3710	3710	5082	5908	2648	2648	4592	7440
#48	1228	3680	3680	4908	6142	2454	2454	4906	7360
#49	1302	3665	3681	4967	6034	2550	2546	4826	7396
#50	1231	3688	3678	4899	6145	2452	2462	4894	7366
#51	1468	3555	3555	4845	6240	2392	2392	4892	7456
#52	1240	3672	3682	4902	6148	2456	2446	4900	7366
#53	1212	3671	3675	4887	6116	2472	2466	4920	7386
#54	1275	3672	3702	4927	6109	2412	2452	4952	7344
#55	1221	3681	3665	4886	6115	2456	2460	4916	7396
#56	1221	3671	3675	4896	6107	2472	2466	4920	7386
#57	1242	3709	3667	4857	6116	2508	2446	4936	7370
#58	1275	3702	3672	4927	6109	2452	2412	4952	7344
#59	1227	3687	3665	4880	6115	2474	2462	4936	7376
#60	1225	3680	3680	4905	6145	2460	2460	4900	7360
#61	1240	3682	3672	4902	6148	2446	2456	4900	7366
#62	1216	3679	3671	4883	6116	2474	2466	4922	7384
#63	1321	3710	3710	4881	6109	2384	2384	4856	7440
#64	1221	3665	3681	4886	6115	2460	2456	4916	7396
#65	1212	3671	3675	4887	6116	2472	2466	4920	7386
#66	1221	3681	3665	4886	6115	2456	2460	4916	7396
#67	1275	3702	3672	4927	6109	2452	2412	4952	7344
#68	1221	3665	3681	4886	6115	2460	2456	4916	7396
#69	1221	3681	3665	4886	6115	2456	2460	4916	7396

For example, the projection matrix in table 1-E_8 labelled as #26 belongs to the A_1(principal). The branching rule (2*a*) for the three representations of figure 1 is given on line #26 of table 4-E_8. One finds the branching rules

$$(10000000) \supset 14\Gamma_2 \oplus 14\Gamma_3 \oplus 16\Gamma_4 \oplus 20\Gamma_5$$

$$(00000010) \supset 75\Gamma_1 \oplus 185\Gamma_2 \oplus 185\Gamma_3 \oplus 260\Gamma_4 330\Gamma_5$$

$$(00000001) \supset 2455\Gamma_1 \oplus 7350\Gamma_2 \oplus 7350\Gamma_3 \oplus 9805\Gamma_4 \oplus 12\,295\Gamma_5 .$$

Inspecting other lines of the table, one finds that the branching rules above are repeated not only for #25, which is the case (2*b*) of A_1(subprincipal), but also for #24, #23, #22, #19, #10.

The coinciding branching rules are summarized in table 5. Note that #22 and #23 of E_7 do not appear together in the table. Hence the conjecture is ruled out in this case.

Table 5. The numbers on the same line of a cell identify the coinciding branching rules. Each branching rule is specified by its number # from table 4. The cases of the principal and subprincipal A_1 are respectively #9 and #8 in E_6, #23 and #22 in E_7, #26 and #25 in E_8.

$E_6 \supset T_6$ #	$E_7 \supset T_7$ #	$E_8 \supset T_8$ #
1, 5	6, 16, 20	9, 18
3, 6, 7, 8, 9	7, 17, 19, 21, 23	10, 19, 22, 23, 24, 25, 26
14, 19	13, 18	13, 20
15, 18	14, 22	16, 21
17, 20	34, 42	45, 62
	35, 43, 44	53, 65
		55, 66, 69
		58, 67
		64, 68

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